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THE JOURNAL

OF THE

Indian Mathematical Society.

Vol. III.] FEBRUARY 1911. [No. 1.

PROGRESS REPORT.

The following gentlemen have been elected by the Committee as members of the Society.

1. Mr. Barjerji E. Vachha B. Sc., L.C.E., Executive Engineer, P.W.D., Larkhana (Sind).
2. Mr. S. Appukutan Erady, B.A., M.A. Student, 68, Office Venkatachala Mudali Street, Triplicane, Madras (*at concessional rate*).
3. Mr. K. R. Subbarayar, B.A., L.T., Mathematics Teacher, Coimbatore College, Coimbatore (*at concessional rate.*)
4. Mr. A. R. Gopaliah, B. A., L.T., Sub-Assistant Inspector of Schools, Tirukoilur (*at concessional rate.*)
5. Mr. Lallubhai A. Shal, M.A., LL.B., High Court Pleader, 3, Parsivada, Girgaum, Bombay.
6. Mr. V. T. Srinivasa Aiyengar, B.A., B.C.E., Assistant Engineer, 23, Sivaraman Street, Triplicane, Madras.
7. Mr. N. Sankara Aiyar, B.A., M.A. Student, Pycroft's Road, Triplicane, Madras (*at concessional rate.*)

2. The following book has been received as a present to our Library—

"A Text book on Integral Calculus" by Prof. Ganesh Prasad.

3. The Draft of Revised Constitution was submitted to the General Body for vote along with our December Journal. The Committee cannot but feel gratified to find that all those members who voted on the proposal, voted unanimously "for" it. Accordingly the old constitution will be taken as suspended, and the New Constitution brought into force from the date of issue of this report. The first effect of the new Constitution is the change of name from "Club" to "Society," as will be seen in the title of our Journal.

4. A set of Bye-laws for the general working of the Society have also received the assent of the Committee and they will also be considered to be in force from the date of issue of this report. The Bye-laws as well as the Constitution are published below for the information of the General Body.

POONA,
February 1911.

D. D. KAPADIA,
Hon. Joint Secretary.

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Constitution and Bye-laws of the Indian Mathematical Society.

CONSTITUTION.

I. The Society shall be called the Indian Mathematical Society.

II. The objects of the Society shall be the promotion of Mathematical Study and Research in India.

III. Its head quarters shall be at Poona for the time being.

IV. All persons interested in the objects of the Seciety shall be eligible for election as members.

V. Candidates for election must be proposed by one member and seconded by another member of the Society who have personal knowledge of them and their attainments.

VI. (*a*) The annual subscription shall be Rs. 15, payable in advance on the 1st of January. Each new member shall pay in proportion to the unexpired portion of the year at the time of his election.

(*b*) On payment of Rs. 150 in one sum a member shall become a Life-member and shall thereafter be exempt from all future annual subscriptions.

(*c*) The Committee shall have power to reduce the annual subscription of any member to Rs. 10 a year, and to terminate this concession whenever they think fit.

VII. The affairs of the Society shall be managed by a Committee consisting of a President, a Treasurer, a Librarian, two Secretaries and seven additional members. The Committee shall be elected as provided for in rule VIII, and shall have power.

(*a*) to elect new members and to accept resignations of members;

(*b*) to elect as Patrons or as Hon. Members persons distinguished in Mathematics or occupying eminent positions and interested in the objects of the Society;

(*c*) to frame such bye-laws, not inconsistent with the constitution, as may appear necessary;

(*d*) to appoint as many assistants to the Secretaries, the Librarian and the Treasurer, as may be necessary to carry on the work of the Society efficiently.

(*e*) to sanction sums of money for ordinary working expenses, for the purchase of books, journals &c., for printing (under conditions to be determined from time to time by the Committee) papers submitted to the Society, and for such objects *e. g.*, special research in or reports on the teaching of Mathematics etc., as may further the aims of the Society;

(*f*) to sanction the formation of branches of the Society on such terms as appear suitable in each case, and to revoke such sanction;

(*g*) to arrange if considered necessary for registering the Society as a corporate body;

(*h*) to appoint auditors for the audit of accounts;

(*i*) to exempt from payment of subscription any officer of the Society;

and (*j*) generally to deal with matters similar to the foregoing in the best interests of the Society.

VIII. The Committee shall be re-constituted every year as under—

(*a*) The Treasurer and Librarian shall be permanent officers of the Society, to be appointed by the Committee, who may, at their discretion relieve either of them of his appointment.

(*b*) The President and the Secretaries shall always be eligible for re-election, and the Additional Members shall retire two each year in rotation,—this rotation to be determined in the first instance by lot or voluntary retirement, and subsequently by seniority in longest continuous standing coupled with lot, if necessary, in cases of equality in such standing; such retiring members shall not be eligible except as President, for re-election the same year, but may be re-nominated in any subsequent year.

(*c*) The retiring Committee shall propose for election a President and such of its members, as are eligible for re-election and willing to serve together with such new members as are required to constitute a Committee of full strength; other nominations by any member whether for the Presidents, or ordinary membership of the Committee must be sent in within one month after the preliminary nominations of the Committee have been circulated: and in such cases, the election of the President and members of the Committee from all those nominated shall proceed by voting papers. The Committee shall circulate their nominations by the end of August in each year.

(*d*) The Secretaries shall be elected by the new Committee and shall hold office until the next Committee shall have elected its Secretaries.

(*e*) In case of a vacancy in the office of the President or any other member of the Committee owing to resignation or otherwise the remaining members of the Committee shall appoint his successor for the remainder of his term of office.

IX. No decision of the Committee shall be invalid merely on the ground that the office of the President or other officer or member of the Committee was vacant at the time of the decision.

X. The Committee shall publish, not later than the month of April, duly audited accounts for the previous year.

XI. (*a*) The Committee shall submit to the General body any proposal to amend the Constitution that may be accepted by it, and may, if they think fit, so submit any other proposal, whether it relates to the Constitution or not.

(*b*) Any member may thereupon submit any amendments or obje[illegible] for the [illegible]sideration of the Committee.

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(c) All such amendments and objections shall be brought to the notice of the Committee: and the Committee may, with reference thereto, modify their motion in such a manner as may seem to them fit, or they may resubmit their original motion for vote.

(d) The procedure upon a motion made under clause (c) shall be the same as upon an original motion, unless the Committee make it as a final motion.

(e) Where under clause (d) a motion is made as a final motion, the vote recorded by members thereupon, or, upon every part thereof, shall be definitive, that is to say, either in the affirmative or negative.

(f) In all cases of a final motion as well as in case of an original motion, or a motion dealt with as such under clause (d) where no amendments or objections are received for consideration, the motion or any part thereof may, in the discretion of the President be declared "passed by the general body" if a majority of members voting have voted in favour thereof; and it shall be declared to be so passed if a majority of members of the Society have voted in favour thereof.

(g) The Committee may withdraw any motion at any stage.

(h) Provided that no article of Constitution shall be amended unless $\frac{2}{3}$ of the persons voting on the proposal, have voted in favour of it.

Bye-Laws.

I. The Society shall publish a periodical to be called the Journal of the Indian Mathematical Society, as an organ of the Society. Copies of the Journal shall be supplied to the members of the Society free of charge, and to non-members at such rates as shall be fixed from time to time. Authors of papers published in the Journal shall be supplied with any number of copies of their papers at cost price.

II. The Librarian and the Treasurer shall be exempt from the payment of annual subscription as long as they held office.

III. The decisions of the Committee shall be by a majority of votes, the President having a vote and (in case of equality of votes) a casting vote. The President may also, for reasons to be recorded, vote any proposal passed by the Committee and not already approved by him.

IV. A member of the Committee may move an amendment to any proposal before Committee, and such amendments shall be voted upon at the same time as the original proposals, unless the President directs otherwise.

V. It shall be the duty of the *Librarian* to decide, subject to the general directions of the Committee, what books to purchase for the Library. In buying books preference shall be given to such works as are of an advanced character. The Librarian shall also propose every year what Journals he recommends for purchase during the ensuing year.

VI. The *Treasurer* shall receive subscriptions and all sums of money due to the Society. He shall make payments of all the ordinary expenditure provided for by the Committee in the annual budget, but shall receive the sanction of the Committee in the first instance for every extra item of expenditure. He shall keep a regular account of all receipts and payments, and shall submit every quarter a statement of accounts for the information of the Committee. A duly audited Balance sheet shall be prepared by him to be circulated for the information of the General Body as soon as possible before the month of March in each year.

VII. Of the two secretaries—one shall, in general, conduct the correspondence, and the proceedings of the Society and the Committee and shall keep a record of the same: he shall also submit the names of candidates proposed for election, shall prepare an annual budget in conjunction with the Treasurer, shall publish periodical reports of the progress of the society for the information of the General Body, and in general shall perform all miscellaneous work in connection with the society. While the other shall direct and supervise the work of printing and publishing regularly the journal under conditions to be determined from time to time by the Committee with the aid of the Editors and such assistants as may be appointed from time to time by the Committee.

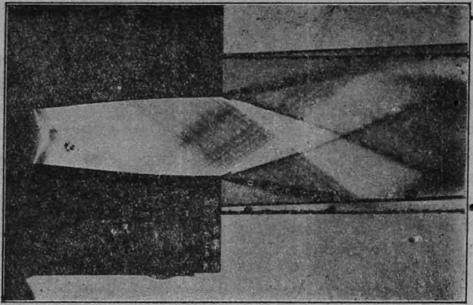
VIII. The Assistant Officers of the society shall attend to such duties as shall be entrusted to them from time to time by the Committee.

On the motion of a gas in two dimension.

By A. Steichen, S. J. Dr. Phil, St. Xavier's College, Bombay.

Introduction.

Using an arrangement devised by Tœpler[1], we get under certain circumstances[2] the following photograph of a jet of gas passing through a mouthpiece :



A system of lenses etc., is combined in such a way, that the portions in the jet where there is expansion appear white, and such parts where there is compression appear dark. The mouthpiece is constructed so that the stream-lines are parallel when leaving the opening. The experiment is arranged so that the velocity of the gas in some parts of the mouthpiece is greater than the velocity of sound.[3] Within the mouthpiece the pressure of the gas is greater than the pressure of the outer air. Hence the jet must expand, when leaving the mouthpiece. There is, therefore, a disturbance caused in the jet at the edges of the opening. From physics it is known that the velo-

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1 Oswald's Klassiker N. 157 and 158.
Kohlrausch, Prakt. Phys. ed. 11 p. 291.

2 Phys. Zeitschr. vol. 8 p. 26.

3 In the narrowest part of the mouthpiece the velocity of the gas is equal to the velocity of sound. After this the velocity increases as the opening becomes wider.

city of propagation of any disturbance in a gas is equal to the velocity of sound corresponding to the pressure p of the gas. Since the gas is moving with a velocity greater than that of sound, the disturbance caused at the edges cannot be felt in the jet to the left of a certain radius-vector r_1 fig 1

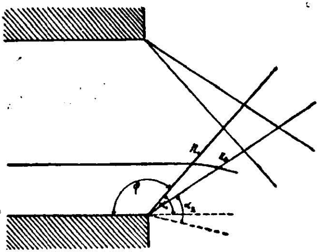

Fig 1.

r_1 forms with the direction of the stream-lines on r_1 the angle α_1 defined by the equation

$$\sin \alpha_1 = \frac{c_1}{w_1}$$

where c_1 = velocity of sound on r_1

w_1 = velocity of the gas on r_1

The angle α is called Mach's angle. The expansion of the gas in the jet will last until the pressure in the gas is equal to that of the outer air. Suppose r_2 to be the radius-vector along which this equilibrium of pressure is obtained. In the region of expansion there is a deflection of the stream-lines. The stream-lines on r_2 make with this radius-vector an angle α_2 defined by the equation

$$\sin \alpha_2 = \frac{c_2}{w_2}$$

where c_2 = velocity of sound on r_2

w_2 = velocity of the gas on r_2

$w_2 > w_1$ on account of the expansion.

$c = c(p)$ if we assume

$pv^k = \text{const.}$

i.e., the adiabatic law. In the whole region of expansion $p = \text{const.}$ and $w = \text{const.}$ on the same radius-vector. Hence p and w are functions of the angle ϕ only.

The two regions of expansion cross in the middle of the jet. The object of this paper is to give a mathematical treatment of the motion of the gas in the regions of expansion, including that portion where the two regions coming from either corner cross. The motion of the gas is approximately a two—dimensional motion in the special case where the mouthpiece is constructed, so that in the direction of the axis, plane parallel sections are equal in form and magnitude, We consider this case only,

Mathematical treatment of the subject.

§ 1. Notations and restrictions.

We shall use the following notations :

p = pressure of the gas in the jet

p_0 = pressure of the gas in the vessel from which it streams forth.

S = density of the gas in the jet.

S_0 = density of the gas in the vessel.

c = velocity of sound. c is a function of p.

c' = critical velocity of sound *i.e.* velocity of sound in that part of the jet where the velocity of the gas is equal to the velocity of sound.

w = velocity of the gas.

ξ and η = components of w // to the axes of x and y.

u and v = components of w // and $\perp$ to the radius-vector in polar coordinates.

α = Mach's angle, defined by the equation

$$\sin\ \alpha = \frac{c}{w}$$

$K = \frac{c_p}{c_v} = 1{\cdot}405$ for air.

$C = \frac{k}{k-1}\frac{p_0}{S_0}$ = the constant of the equation of energy.

t = time.

As to the motion of the gas we make the following restrictions :

(*a*) There shall be no friction and no conduction of heat so that S is a function of p only. (Homogeneous gas)

(*b*) There shall be no external forces.

(*c*) $p = aS^k$ (adiabatic law)

Where a = const.

The motion of the gas shall be

(*d*) two-dimensional,

(*e*) without vortices,

(*f*) steady, (*g*) *w7c*.

§ 2. The main equations.

We have at our disposal the three following equations:

(*a*) For a twodimensional motion of a gas the equation of continuity takes the form:

$$\frac{\partial(S\xi)}{\partial x}+\frac{\partial(S\eta)}{\partial y}+\frac{\partial S}{\partial t}=0 \quad \dots \quad \dots (1)$$

(*b*) There are no vortices

$$\therefore \frac{\partial \xi}{\partial y}-\frac{\partial \eta}{\partial x}=0 \quad \dots \quad \dots \quad \dots (2)$$

(*c*) The third equation is the equation of energy[1]:

$$\int\frac{dp}{S}+\frac{w^2}{2}=C \quad \dots \quad \dots \quad \dots \quad \dots (3)$$

§ 3. The regions of simple expansion.[2]

We examine first the motion of the gas in the two quasi-triangular regions, where there is only simple expansion. On account of the symmetry, it will suffice to consider only one of the two regions. The state of the gas, as we have already pointed out, is a function of a certain angle only, along any radius-vector *r* it is unchanged. We therefore transform the three equations to polar co-ordinates, taking the pole on the edge of the mouthpiece. From fig. 2 we get:

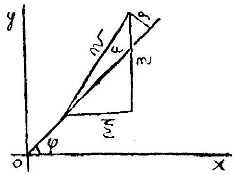

Fig. 2.

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[1] Euzykl der math. Wiss. V. 5b 16 . 17.

[2] This § forms part of the dissertation of Dr. Ph. Meyer Gottingen 1908.

$$\xi = u \cos \phi - v \sin \phi$$

$$\eta = u \sin \phi + v \cos \phi$$

In these co-ordinates the three equations of § 2 take the following forms:

$$v\frac{dS}{d\phi} + S\left(u + \frac{dv}{d\phi}\right) = 0 \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$\frac{du}{d\phi} - v = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$\int \frac{dp}{S} + \frac{u^2 + v^2}{2} = C \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

(*a*) Discussion of these equations.

If c = velocity of sound corresponding to p and S then

$$c^2 = \frac{dp}{dS}$$

∴ equation (3) may also be written

$$\int \frac{c^2 dS}{S} + \frac{u^2 + v^2}{2} = C \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

Differentiating with respect to ϕ we get:

$$\frac{c^2 dS}{S d\phi} + \left[u\frac{du}{d\phi} + v\frac{dv}{d\phi}\right] = 0 \quad \ldots \quad (3a)$$

From equation (2) we get:

$$\frac{du}{d\phi} = v$$

∴ from (3*a*)

$$\frac{1}{S}\frac{dS}{d\phi} = -\frac{v\left(u + \frac{dv}{d\phi}\right)}{c^2} \quad \ldots \quad \ldots \quad (3')$$

and from equation (1)

$$\frac{1}{S}\frac{dS}{d\phi} = -\frac{u + \frac{dv}{d\phi}}{v} \quad \ldots \quad \ldots \quad (1)$$

From equations (2) and (3′) it follows.

$$\left[u + \frac{dv}{d\phi}\right](v^2 - c^2) = 0$$

This equation admits of two solutions:

1st Solution.

$$u + \frac{dv}{d\phi} = 0$$

[1] Helmholtz, Vorlesungen uber theor. Phys, II §. 58.

Hence it follows from (1)

$$\frac{dS}{d\phi}=o \therefore S=\text{const.}$$

Combining the two equations

$$u+\frac{dv}{d\phi}=0 \text{ and } \frac{du}{d\phi}-v=0$$

we get the integrals

$$u=A_1 \cos \phi+A_2 \sin \phi$$
$$v=-A_1 \sin \phi+A_2 \cos \phi$$

$$\therefore w=\sqrt{u^2+v^2}=\sqrt{A_1^2+A_2^2}$$

i.e. a constant velocity.

By substituting the above values of u and v into the equations.

$$\xi=u \cos \phi-v \sin \phi$$
$$\eta=u \sin \phi+v \cos \phi$$

we get

$$\xi=A_1 \quad \eta=A_2$$

this means a parallel constant motion of the gas. Hence we find that in a constant parallel motion of the gas the density is constant.

2nd Solution.

$$v=c$$

This gives us the important result that the component of velocity perpendicular to the radius-vector, is equal to the velocity of sound corresponding to the state of the gas at the given point.

(*b*) Solution of the problem.

(*a*) The equation of the stream-lines.

We assume the adiabatic law.

$$p=aS^k$$
$$\therefore dp=akS^{k-1}dS$$

$\therefore$ equation (3) takes now the form

$$\int akS^{k-2}dS+\frac{u^2+v^2}{2}=C \qquad (3)$$

$$\therefore \frac{ak}{k-1}S^{k-1}+\frac{u^2+v^2}{2}C=$$

Also

$$c^2=\frac{dp}{dS}=akS^{k-1}=v^2$$

Hence we get from equation (2)

$$\left(\frac{du}{d\phi}\right)^2+\frac{k-1}{k+1}u^2=\frac{2(k-1)}{k+1}C$$

$$\therefore d\phi = \frac{du}{\sqrt{\frac{2(k-1)}{k+1}C - \frac{k-1}{k+1}u^2}}$$

$$\therefore u = \pm\sqrt{2C}\sin\left[(\phi+B)\sqrt{\frac{k-1}{k+1}}\right]$$

and from (2).

$$v = \sqrt{2C}\sqrt{\frac{k-1}{k+1}}\cos\left[(\phi+B)\sqrt{\frac{k-1}{k+1}}\right]$$

B is an arbitrary constant. If we reckon ϕ so that $u=o$ for $\phi=o$ then $B=o$ and we get:

$$u = \sqrt{2C}\sin\left(\phi\sqrt{\frac{k-1}{k+1}}\right)$$

$$v = \sqrt{2C}\sqrt{\frac{k-1}{k+1}}\cos\left(\phi\sqrt{\frac{k-1}{k+1}}\right)$$

These are the two components of velocity.

Hence we get the equation of the stream-lines in polar co-ordinates

$$\frac{d\,lgr}{d\phi} = \frac{u}{v} = \sqrt{\frac{k+1}{k-1}}\tan\left(\phi\sqrt{\frac{k-1}{k+1}}\right)$$

$$\therefore r = r_o\left[\cos\left(\phi\sqrt{\frac{k-1}{k+1}}\right)\right]^{-\frac{k+1}{k-1}}$$

r_o being the length of the radiusvector for $\phi=o$

r goes to ∞ for $\phi = \frac{\pi}{2}\sqrt{\frac{k+1}{k-1}} = 219^\circ\ 19'$

This is the limiting vlaue of ϕ when the gas streams into the vacuum.

(β.) Deviations of the jet for different valves $\frac{p}{p_o}$.

Suppose.

p_o = pressure of the gas in the vessel from which it streams forth.

S_o = density of the gas in the vessel.

p_1 = pressure of the gas on the edge of the mouthpiece.

S_1 = density of the gas on the edge of the mouthpiece.

p_2 = pressure of the outer air

S_2 = density of the outer air

We found already

$$w^2 = u^2+v^2 = 2\,C\left[\sin^2\left(\phi\sqrt{\frac{k-1}{k+1}}\right) + \frac{k-1}{k+1}\cos^2\left(\phi\sqrt{\frac{k-1}{k+1}}\right)\right]$$

$$= \frac{2\,C}{k+1}\left[k - \cos\left(2\,\phi\sqrt{\frac{k-1}{k+1}}\right)\right]$$

From equation of energy[1] we get

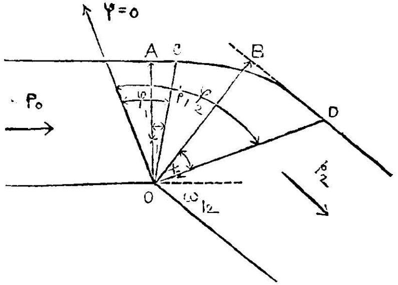
$$w^2=\frac{2k}{k-1}\frac{p_0}{S_0}\left[\lambda-\left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}\right]$$

and $C=\frac{k}{k-1}\frac{p_c}{S_0}$

$$\therefore \phi=\tfrac{1}{2}\sqrt{\frac{k+1}{k-1}}\cos^{-1}\left[(k+1)\left(\frac{p}{p_0}\right)^{\frac{k-1}{k}}-1\right]\dots\quad(4)$$

Equation (4) gives the relation between ϕ and $\frac{p}{p_0}$. We must now give a fuller discussion of the expansion.

We suppose that in the mouthpiece the stream lines are parallel, and that the velocity of the gas is greater than the velocity of sound.



Suppose O to be the edge of the mouthpiece. The disturbance caused at the edge is propagated along a radius-vector OC, making an angle α_1 (Mach's angle) with the stream-lines. Call ψ_1 the complementary angle of α_1. Along OC the expansion begins and goes on until on a certain radius-vector OD p becomes equal to p_2. After the expansion there will be again a parallel motion of the gas. Call ψ_2 the complementary angle of the Mach's angle which OD makes with the direction of the new stream-lines.

We reckoned ϕ from the radius-vector perpendicular to the stream-lines in the mouthpiece. From fig. 3 we get

$$\tan\psi_1=\frac{u}{v}=\sqrt{\frac{k+1}{k-1}}\tan\left(\phi\sqrt{\frac{k-1}{k+1}}\right)\quad\dots\quad(5)$$

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[1] Enzyk l. der math. Wiss. V. 5 b. 18.

∴ For small values of ϕ_1 we have approximately

$$\phi_1 = \psi_1$$

and for $\psi = 0$ $\phi = 0$ and $u = 0$

i.e., the radius vector perpendicular to the parallel motion coincides with the radius-vector, making an angle $\phi = o$ in our co-ordinates, as was to be expected.

$$w = \sqrt{u^2 + v^2}$$

∴ for $u = o$ $w = v$ and $\phi = 0$

∴ $\phi = o$ where the velolc ty of the gas is equal to the velocity of sound.

Generally we get for $\psi > o$, $\phi > \psi$.

∴ the direction $\phi = o$ is to be taken as in fig. 3.

If we know p_0, p_1 and p_2 the two equations.

$$\phi = \tfrac{1}{2}\sqrt{\frac{k+1}{k-1}} \cos^{-1} \left[(k+1) \left(\frac{p}{p_0}\right)^{\frac{k-1}{k}} - 1 \right] \qquad (4)$$

$$\tan \psi = \sqrt{\frac{k+1}{k-1}} \tan \left(\phi \sqrt{\frac{k-1}{k+1}} \right) \qquad (5)$$

will enable us to find the deviation of the jet. Call the angle of deviation w_{12} then from fig. 3 we get immediately.

$$w_{12} = (\phi_2 - \psi_2) - (\phi_1 - \psi_1) = v_2 - v_1$$

ϕ_1, ϕ_2, ψ_1, ψ_2, we get from (4) and (5) if we know p_0, p_1 and p_2. Vice versa if we know $p_0\, p_1$ and w_{12} we can find p_2.

From (4) we get by ordinary trigonometry

$$\tan \left(\phi \sqrt{\frac{k-1}{k+1}} \right) = \sqrt{\frac{2}{k+1} \left(\frac{p_0}{p} \right)^{\frac{k-1}{k}} - 1} \quad \ldots \quad (6)$$

an equation which is more convenient for numerical calculations than equation (4). Moreover we see immediately from the equation in its new form that

$$0 \leqq \frac{p}{p_0} \leqq \left(\frac{2}{k+1} \right)^{\frac{k}{k-1}}$$

where $\frac{p}{p_0} = 0$ means that the gas streams from the vessel into the vacuum.

For $\frac{p}{p_0} = \left[\frac{2}{k+1} \right]^{\frac{k}{k-1}} = 0{\cdot}527$ (for $k =$ and $1{\cdot}405$) we get $\phi = o \psi = o$ $u = o$ $w = v = c$ *i.e.* the value of ϕ, where the velocity of the gas is equal to the velocity of sound.

On the Impossibility of the Indeterminate Equation $x^n+y^n=nz^n$, in which n is an odd prime integer.

By T. Hayashi, Tokyo, Japan.

[1.] Since Lebesgue's* and Dirichlet's† studies on the indeterminate equation

$$x^5+g^5=az^5,$$

where a is any integer, several investigators have tried to solve the indeterminate equations of the form

$$x^n+y^n=az^n.$$

Among others, the memoirs by Ed. Maillet‡ are especially worthy of notice. He has given four categories of the equation which are impossible in integers. The question of the impossibility of the equations of such a form is fast becoming as classic as that of the last theorem of Fermat's.

In this note§, I will treat the impossibility of the indeterminate equation

$$x^n+y^n=nz^n,$$

where n is an odd prime integer, and give a category of the equations which are also impossible in integers. The conditions to specify the category are given by the sums of the integral coefficients of the two remarkable polynomials Y and Z in ξ satisfying the identical relation

$$4\cdot\frac{\xi^n-1}{\xi-1}=Y^2-(-1)^{\frac{n-1}{2}}nZ^2,*$$

when n is an odd prime integer.

[2.] Let the equation

(1.) $$x^n+y^n=nz^n,$$

be possible, when n is an odd prime number.

First of all, we can suppose that x, y, z have no common factor.

Since $(x+y)^n-(x^n+y^n)$ is a multiple of n, $x+y$ is also a multiple of n. Let

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* J. Math. pures appl. (1) 8 (1843), p. 49.

† J. reine angew Math. 111 (1893), p. 26.

‡ C. R. Acad. sc. Paris 129 (1899), p. 198 ; 140 (1905), p. 1229 ; Acta Math 24 (190), p. 2471 ; Utcm. Acad sc. Toulouse (10) 5 (1905), p. 1321 ; Ann. Mat. pura appl. (3) 12 (1905), p. 145.

§ For the first part of this note, I am indebted to Mr. Jamshedji Edalji, late professor of Mathematics in Gujarat College, Ahmedabad, India.

* Cauchy, Oeuvres (1) 5. p. 85 ; Gauss, Werke 1, p. 443, etc.

(2.) $x+y=nk.$

Then

(3.) $x^n+(nk-x)^n=nz^n.$

Expanding the second term, we get

(4.) $n^nk^n-n^nk^{n-1}x+\binom{x}{2}n^{n-2}k^{n-2}x^2-\ldots\ldots\ldots\ldots$

$$-\binom{n}{2}n^2k^2x^{n-2}+n^2kx^{n-1}=nz^n.$$

Since the left hand member is divisible by n^2k, the right hand member must be so also. Hence z is divisible by n. Let

(5) $z=n^p\,.\,h,$

where p is an integer $\geqq 1$, and h is an integer prime to n.

Since x, y, z are supposed to have no common factor and z is divisible by n, x and y are prime to n. Hence x and y have no common factor.

By (4.)

(a multiple of n^3k^2)$+n^2kx^{n-1}=nz^n$,

or

(6.) $n^2k\,\{$ (a multiple of nk)$+x^{n-1}\}=nz^n.$

Hence if we put

(7.) $nz^n=n^2kk',$

then k' is prime to n, because x is prime to n.

By (5).

$n^{np+1}h^n=n^2kk'.$

Hence k is divisible by n^{np-1}. Let

(8.) $k=n^{np-1}l,$

where l is prime to n. Then

(9) $h^n=lk'.$

If we assume that k and x have a common factor a, then a must be a factor of y by virtue of (2), and must be a factor of z by virtue of (7); so that x, y, z have a common factor, which is contrary to our first supposition. Hence k and x have no common factor.

Hence by (6) and (7), k' which stands for (a multiple of nk) $+x^{n-1}$ is prime to k. Hence by (8.), l and k' are prime to each other. Hence, by (9.), we can put

(10.) $l=l_1{}^n,\ k^1=k_1{}^n,$

l_1 and k_1 being prime to each other, and both being prime to n (respectively).

Therefore we arrive at the following equalities:

$$(11.)\begin{cases} x^n+y^n=n(x+y)k_1^n \\ \qquad =n\odot nk\cdot k_1^n \\ \qquad =n\odot n^{np}l_1^n\odot k_1^n. \end{cases}$$

[**3.**] Now let us consider the well-known theorems · if n be an odd prime number, we can find two polynomials Y and Z having integral coefficients which satisfy the identity

$$4.\ \frac{\xi^n-1}{\xi-1}=Y^2-(-1)^{\frac{n-1}{2}}nZ^2,$$

the degree of Y with respect to ξ being $\frac{n-1}{2}$, and that of Z being $\frac{n-1}{2}-1$

Let

$$\nu=\frac{n-1}{2},$$

$$Y=b_0\xi^{\nu}+b_1\xi^{\nu-1}+b_2\xi^{\nu-2}+\dots\dots+b_{\nu-1}\xi+b_{\nu}$$

$$Z=\qquad c_0\xi^{\nu-1}+c_1\xi^{\nu-2}+\dots\dots+c_{\nu-2}\xi+c_{\nu-1}.$$

Write $\frac{y}{x}$ for ξ, and then change the sign of x; we get

$$(12.)\qquad 4.\ \frac{x^n+y^n}{x+y}=\eta^2-(-1)^{\nu}n\,(x\zeta)^2,$$

where

$$13.\begin{cases} \eta=b_0y^{\nu}-b_1y^{\nu-1}x+b_2y^{\nu-2}x^2-\dots \\ \qquad +(-1)^{\nu-1}b_{\nu-1}yx^{\nu-1}+(-1)^{\nu}b_{\nu}x^{\nu} \\ \zeta=\qquad c_0y^{\nu-1}-c_1y^{\nu-2}x+c_2y^{\nu-3}x^2-+\dots\dots+ \\ \qquad (-1)^{\eta-1}c_{\nu-1}x^{\nu-1}. \end{cases}$$

[**4.**] Connecting the results of the two preceding sections, we must have

$$4\,n\,k_1^n=\eta^2-(-1)^{\nu}n(x\zeta)^2,$$

when our indeterminate equation is possible.

Multiply the two members of this equation by n^{n-1}; then

$$(14.)\quad 4\,(n\,k_1)^n=(n^{\nu}\eta)^2-(-1)^{\nu}n(n^{\nu}x\zeta)^2.$$

This equation is satisfied by

(15) $uk_1 = s^2 - Dt^2$,

(16.) $$n^\nu \eta = 2\left[s^n + \binom{n}{2} s^{n-2} t^2 D + \binom{n}{4} s^{n-4} t^4 D^2 + \dots\dots + nst^{n-1} D^{\frac{n-1}{2}} \right],$$

$$n^\nu x \zeta = 2\left[ns^{n-1} t + \binom{n}{3} s^{n-5} t^3 D + \binom{n}{5} s^{n-5} t^5 D^2 + \dots\dots + t^n D^{\frac{n-1}{2}} \right],$$

where D stands for $(-1)^\nu n$, and s and t are any integers whatever.

If the divisors of the right hand member of (14.) be 2, n, or else have the form $s^2 - (-1)^\nu n t^2$, then (15.) gives us the general solution of equation (14.), or nk_2 must have the form $s^2 - Dt^2$.

The right hand member of (14.) is

$$n^{n-1} \{ \eta^2 - (-1)^\nu n (x\zeta)^2 \} ;$$

and since the two polynomials Y and Z, and therefore η and $x\zeta$ have been calculated for several values of n (*e. g.* see Gauss, Werke 1, p. 443; Legendre, Zahlen theorie, deutsch von Maser, Bd. II, p. 189; Legendre, Mémoire sur la determination des fonctions Y et Z qui satisfont a l'equation 4 $(x^n - 1) = (x-1)(Y^2 \pm nZ^2)$, n étant un nombre premier $(4i \pm 1)$, Paris, Mém. Acad. Sc. XII (1833), p. 81; Mathews, Theory of Numbers, part, 1, (1892) p. 218), it is easy to determine whether η and $x\zeta$ have a common divisor or not, by such a process as in article 6.

If we find that η and $x\zeta$ have no common factor, then we can find for n such a value that the binary quadratic form $\eta^2 - (-1)^\nu n (x\zeta)^2$ has the divisors of the form $s^2 - (-1)^\nu nt^2$ alone, by using the tables of divisors of binary quadratic forms (*e.g.* Legendre, Zahlenthéorie, deutsch von Maser, Bd. I, p. 394. Tafel III; p. 407, Tafel V).

If we find that η and $x\zeta$ have a common factor, we select only the cases where the common factor is 2*. Then again we can find for n such a value that the binary quadratic form $\eta^2 - (-1)^\nu n (x\zeta)^2$ divided by the square of the common factor of η and $x\zeta$, has the divisors of the form $s^2 - (-1)^\nu nt^2$ alone, by using the tables of divisors of binary quadratic forms.

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* In the cases I examined, the common factor, if it existed, was always 2, as in the examples in article 6.

By (15.), s must be divisible by n, since D is divisible by n.

Hence all the terms of the right hand member of (16.) are divisible by $n^{\nu+2}$, and consequently its left hand member must be so also.

Hence η must be divisible by n^2.

[5.] Now let us find the quotient of η divided by $x+y$ by means of equation (13.); then we get

$$\eta = \text{a multiple of } (x+y)+(-1)^{\nu}\,(b_0+b_1+b_2+\ldots\ldots+b_{\nu})x^{\nu}.$$

But in our case, by (11.),

$$x+y=n^{np}l_1^{\,n},$$

p being an integer $\geqq 1$.

Hence when η is divisible by n^2, the sum $b_0+b_1+b_2+\ .\ \ldots+b_{\nu}$ must be divisible by n^2, since x is prime to n.

Therefore we arrive at the following result:

A necessary condition for the possibility of the indeterminate equation

$$x^n+y^n=nz^n$$

for an odd prime n, is

$$b_0+b_1+b_2+\ldots\ldots+b_{\frac{n-1}{2}}\equiv 0, \qquad (mod.\ n^2)$$

where $b_0, b_1, b_2, \ldots\ldots, b_{\frac{n-1}{2}}$ are the coefficients of the rational integral function Y satisfying the identity

$$4.\ \frac{\xi^n-1}{\xi-1}=Y^2-(-1)^{\frac{n-1}{2}}nZ^2,$$

$$\textit{where } Y=b_0\xi^{\frac{n-1}{2}}+b_1\xi^{\frac{n-1}{2}-1}+b_2\xi^{\frac{n-1}{2}-2}+\ldots\ldots+b_{\frac{n-1}{2}},$$

$$\textit{and } Z=c_0\xi^{\frac{n-1}{2}-1}+c_1\xi^{\frac{n-1}{2}-2}+c_2\xi^{\frac{n-1}{2}-3}+\ldots\ldots+c_{\frac{n-1}{2}-1},$$

when the two expressions

$$\eta=b_0y^{\frac{n-1}{2}}-b_1y^{\frac{n-1}{2}-1}x+b_2y^{\frac{n-1}{2}-2}x^2-+\ldots\ldots+(-1)^{\frac{n-1}{2}}b_{\frac{n-1}{2}}x^{\frac{n-1}{2}}.$$

$$\textit{and } x\zeta = c_0 y^{\frac{n-1}{2}-1} x - c_1 y^{\frac{n-1}{2}-2} x^2 + - \dots + (-1)^{\frac{n-1}{2}-1} C_{\frac{n-1}{2}-1} x^{\frac{n-1}{2}-1}$$

have such a property that the quadratic form $\eta^2 - (-1)^{\frac{n-1}{2}} n (x\zeta)^2$ *admits only* 2 *and the numbers of the form* $s^2 - (-1)^{\frac{n-1}{2}} nt^2$ *as its divisors.*

[6.] Applications. Let me apply the above theorem for the case $n = 4m+1$.

(i) $m=1,\ n=5$

$Y = 2\xi^2 + \xi + 2, \qquad Z = \xi;$

$\eta = 2y^2 - yx + 2x^2, \qquad x\zeta = yx.$

If η and $x\zeta$ have a common divisor α, it is a divisor of yx and therefore of y or of x.

If α is a divisor of y, it is also a divisor of $2x^2$, since it is a divisor of η. But α is not a divisor of x, for, if α is a common divisor of y and x, it must be a divisor of nz^n; but x, y and z have no common factor; hence we must have $\alpha = n$; but x and y are prime to n as has been shown in article 2. Therefore $\alpha = 2$.

If α is a divisor of x, then we must have also $\alpha = 2$ by the same reasoning.

Therefore if η and $x\zeta$ have a common divisor, it is 2. In this case $\left(\frac{\eta}{2}\right)^2 - 5\left(\frac{x\zeta}{2}\right)^2$ has the quadratic form $s^2 - 5t^2$ only as its divisor (see Legendre, Bd. I, p. 394, Tafel III).

If η and $x\zeta$ have no common factor, then $\eta^2 - 5(x\zeta)^2$ has divisors of the quadratic form $s^2 - 5t^2$ only as its divisors.

Now $b_0 + b_1 + b_2 = 2 + 1 + 2 = 5 \not\equiv 0 \pmod{5^2}$. Therefore the equation

$$x^5 + y^5 = 5z^5$$

is impossible.

(ii) $m=3,\ n=13$.

$Y = 2\xi^6 + \xi^5 + 4\xi^4 - \xi^3 + 4\xi^2 + \xi + 2,\ Z = \xi^5 + \xi^3 + \xi.$

$\eta = 2y^6 - y^5x + 4y^4x^2 + y^3x^3 + 4y^2x^4 - yx^5 + 2x^6,$

$x\zeta = y^5x - y^3x^3 + yx^5.$

If η and $x\zeta$ have a common factor α, it is a divisor of x or of $y^5 - y^3x^2 + yx^4$.

If α is a divisor of x, it is also a divisor of $2y^6$ since it is a divisor of η. But α is not a divisor of y; for, if α is a common divisor of x and y, it must be divisor of nz^n; but x, y and z have no common factor; hence we must have $\alpha = n$; but x and y are prime to n as has been shown in article 2. Therefore $\alpha = 2$.

If α is a divisor of $y^5 - y^2x^3 + y\,x^4$, it is a divisor of

$$\eta - (2y-x)(y^5+y^3x^2+yx^4),$$

i.e. $6y^4x^2+2y^2x^4+2x^6$.

Hence α is a divisor either of 2, or of x, or of $3y^4+y^2x^2+x^4$. If α is a divisor of 2, $\alpha = 2$. If α is a divisor of x, we must have $\alpha = 2$. as before. If α is a divisor of $3y^4+y^2x^2+x^4$, it is a divisor of

$$3(y^5-y^3x^2+yx^4)-y(3y^4+y^2x^2+x^4),$$

i.e. $-4y^3x^2+2yx^4$.

Hence α is a divisor either of 2, of x, of y, or of $-y^2+x^2$. If α is a divisor of 2, $\alpha = 2$. If α is a divisor of x, we must have $\alpha = 2$ as before. If α is a divisor of y, it is a divisor of $2x^6$, since α is a divisor of η; but y and x can have no common factor; hence $\alpha = 2$. If α is a divisor of $-2\,y^2+x^2$ it is also a divisor of $2\,(3\,y^4+y^2\,x^2+x^4)+3\,y\,(-2\,y^2+x^2)$,

i.e. $5\,x^2\,y^2+2\,x^4$.

Hence α is a divisor either of x or of $5\,y^2+2\,x^2$. If α is a divisor of x, we must have $\alpha = 2$ as before. If α is a divisor of $5\,y^2+2\,x^2$, it is also a divisor of $2\,(5\,y^2+2\,x^2)+5\,(-2\,y^2+x^2)$,

i.e. $9\,x^2$.

Hence α is a divisor of x or 9. If α is a divisor of x, we must have $\alpha = 2$ as before. If α is a divisor of 9, $\alpha = 3$ or $\alpha = 9$. But the odd divisor of $-2y^2+x^2$, of which α is a divisor, can not be 3 (see Legendre Bd. I, p. 394, Tafel III); and the odd divisor of $5y^2+2x^2$, of which α is a divisor, can not be 9 (see Legendre, Bd. I, p. 409, Tafel VI).

Therefore we conclude that if η and $x\xi$ have a common factor, it is 2. In this case $\left(\frac{\eta}{2}\right)^2 - 13\left(\frac{x\xi}{2}\right)^2$ has the quadratic form s^2-13t^2 only as its divisors (see Legendore, Bn. I, p. 39*d*, Tafel III).

If η and $x\xi$ have no common factor, $\eta^2 - 13\,(\alpha\xi)^2$ has the quadratic form s^2-13t^2 only as its divisors.

$$\begin{aligned} \text{Now } & b_0+b_1+b_2+b_3+b_4+b_5+b_6 \\ &= 2+1+4-1+4+1+2 \\ &= 13 \\ &\not\equiv 0, \qquad (\text{mod. } 13^2). \end{aligned}$$

Therefore the equation

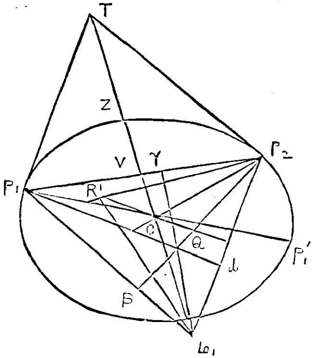
$$x^{13}+y^{13}=13z^{13}$$

is impossible.

SHORT NOTES.

Some examples from Macaulay's Conics.

If C be the centre of a given conic and CK be drawn making with the major-axis an angle equal to the angle made by CQ with it, and a point K be taken on it such that $\frac{CK}{CQ}=\frac{a^2+b^2}{a^2-b^2}$, the rect. hyp. through the feet of the normals from Q to the conic has Q K for a diameter and passes through C, and its asymptotes are parallel to the axes of the given conic This can be very easily proved analytically. The geometrical proof of thistheorem is given in Macaulay's Geometrical conics (page 158.)



1. Let Q P_1, Q P_2 be two normals drawn to the conic and let TP_1, TP_2, be the corresponding tangents. Complete the parm. T P_2 L_1 P_1, Then the above mentiond rect. hyp. passes through L_1. For, L_1 P_1. L_1 P_2 being parallel to T P_2, T P_1 are perpendicular to Q P_2, Q P_1 hence L_1 is the ortho-centre of; the triangle Q P_1 P_2. Hence the rect-hyp. passing through Q, P_1, P_2 passes also through L_1 (Macaulay. Ex. 27 p.134). Similarly for *every, pair of points* like "P" we have a point like-"L," so that there are five other points L_2 L_3 L_4 L_5 L_6 which lie on this rect. hyp.

2. If R_1 be the orthocentre of the $\triangle$ C L_1 P_2. R_1 lies on this rect hyp. Also R_1C is perpendicular to L_1 P_2 *i.e*, parallel to Q P_1. When the positions of P_1 and Q are known, that of R_1 can be found. C R_1 is a chord of the rect. hyp. parallel to the chord Q P_1. The centre O of the rect. hyp. is the middle point of Q K. Hence R_1 can be found from the fact that if O is joined to the middle point of Q P_1, the middle point of C R_1 lies in this line. Similarly we have five other points R_2 R_4 R_3 R_5 R_6 lying on the rect. hyp. Hence the rect. hyp, through the feet of the normals drawn from Q to the conic passes through *twelve* other fixed points.

3. Now TCL_1, is a straight line bisecting P_1 P_2 in V. If P_1 C P_1' be a diameter of the conic, C V is parallel to P_2 P_1', and since C L_1, is

perpendicular to $R_1 P_2$, $P_2 P_1'$ is also perpendicular to $R_1 P_2$. Hence P_2 lies on the circle described on $R_1 P_1'$ as diameter. The positions of R_1 and P_1' depend only on those of P_1 and Q ; therefore P_3 and P_4 also lie on this circle, P_3 and P_4 being the feet of the other normals drawn from Q. Hence if on a normal P_1Q at P_1 a point Q is taken, we have a geometrical construction for finding the other normals that can be drawn from Q ; because R_1 and P_1' are found and the circle on R_1P' as diameter can be described, which cuts the conic in P_2, P_3, P_4. (Macaulay Ex, 28.)

4. From the figure, we can deduce easily that any pair of conjugate diameters of a conic have the same ratio as the parallel diameters of any rect. hyp, whose asymptotes are parallel to the axes of the conic.

Let CV cut the conic in Z. then $CV.\ CT = CZ^2$

$$\therefore\ CV\ .\ VT = CZ^2 - CV^2$$

$$i.e.\ CV\ .\ VL_1 = CZ^2 - CV^2$$

$P_1V^2 : VC\ .\ VL_1 = P_1V^2 : CZ^2 - CV^2 =$ the ratio of the squares of the parallel diameters of the conic. Also CL_1 and P_1P_2 are two chords of the rect. hyp. Hence $\dfrac{VC\ .\ VL_1}{VP_1\ .\ VP_2}$ *i.e.* $\dfrac{VC\ .\ VL_1}{P_1V^2} =$ the ratio of the squares of the parallel diameters of the rect. hyp.

Also C V and $P_1 P_2$ are conjugate chords since C V bisects $P_1 P_2$. Hence, this being true for all positions of Q, the ratio of any two conjugate diameters of a conic is the same as that of the parallel diameters of any rect. hyp. whose asymptotes are parallel to the axes of the conic (Macaulay Ex. 30).

5. If R_1 C produced meets the tangent at P_1' in Y, R_1 C being perpendicular to T P_1 is perpendicular to the tangent at P_1'. Hence the circle on $R_1 P_1'$ as diameter passes through Y. Hence we get the theorem, that any circle through a point (like P_1') on the conic and the foot of the perpendicular from the centre on the tangent at that point cuts the conic again in three points (like P_2 P_3 P_4) the normals at which are concurrent [Macaulay Ex. 29 (i)]. Also the circle on any radius vector (like C P_1') of a conic as diameter cuts the conic again in points the normals at which are concurrent, since this circle passes through the foot of the perpendicular from C on the tangent at the extremity of the radius vector. [Macaulay Ex. 29 (ii)].

R. Srinivasan.

Curvature referred to moving axes.

1. Consider a pair of rectangular axes rotating uniformly about the origin.

Let (u, v) be the co-ordinates of a point referred to the moving axes at any time, and let (X,Y) be the co-ordinates of the same point referred to the initial position of the axes, and let θ be the angle turned through from the initial position.

Then $X = u\cos\theta - v\sin\theta$.

$$Y = u\sin\theta + v\cos\theta.$$

$$\frac{dY}{dX} = \frac{(u'-v)\sin\theta + (u+v')\cos\theta}{(u'-v)\cos\theta - (u+v')\sin\theta}$$

$$\frac{d^2Y}{dX^2} = \frac{(u'-v)(v''+2u'-v) - (u+v')(u''-2v'-u)}{\{(u'-v)\cos\theta - (u+v')\sin\theta\}^2}$$

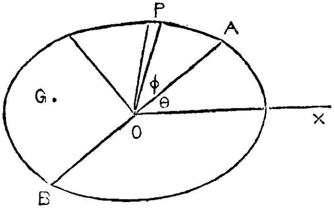
where the dashes refer to differentiation with respect to θ

Hence, after simplification,

$$\frac{1}{\rho} = \frac{(u'-v)(v''+2u'-v) - (u+v')(u''-2v'-u)}{\{(u'-v)^2 + (u+v')^2\}^{\frac{3}{2}}}$$

2. *An application of the above expression.*

An oval curve of area Ω has a centre O, and AB is any chord through O. G is the centroid of one of the two portions of the oval cut off by AB. Show that the radius of curvature at G to the locus of G as AB revolves is $\frac{AB^3}{6\Omega}$, [Oxford, 1906].



Take (u, v) the co-ordinates of G referred to OA and the perpendicular to OA as axes.

Let the oval referred to OX as initial line be $r = f(\theta)$;

then

$$\frac{1}{3}\Omega\, u = \int_0^{\pi} \frac{1}{3} r^3 \cos\phi \, d\phi$$

$$=\frac{1}{3}\int_0^{\pi} \{ f(\phi+\theta) \}^3 \cos\phi \, d\phi$$

$$\frac{1}{2}\Omega\, v=\frac{1}{3}\int_0^{\pi} \{ f(\phi+\theta) \}^3 \sin\phi \, d\phi$$

$$\therefore \frac{3}{2}\Omega\,(u'-v)=\int_0^{\pi} 3\{ f(\phi+\theta) \}^2 f'(\phi+\theta) \cos\phi \, d\phi - \int_0^{\pi} \{ f(\phi+\theta) \}^3 \sin\phi \, d\phi$$

$$=\int_0^{\pi} [\cos\phi \, d \{ f(\phi+\theta) \}^3 + f(\phi+\theta)^3 \, d\cos\phi]$$

$$=\cos\phi \{ f(\phi+\theta) \}^3 \Big]_0^{\pi}$$

$$=-\tfrac{1}{4}\, AB^3 \text{, since } f(\pi+\theta)=f(\theta)=\tfrac{1}{2}\, AB$$

$$\text{Similarly , } \frac{3}{2}\Omega\,(v'+u)=\sin\phi \{ f(\phi+\theta) \}^3 \Big]_0^{\pi}=0$$

This latter merely shows that the tangent at G to the locus of G is parallel to AB.

$$\therefore \frac{1}{\rho}=\pm\frac{v''+2\,u'-v}{(u'-v)^2}=\pm\frac{1}{u'-v}\text{ , sin } v''+u'=0.$$

$$\therefore \rho=\frac{AB^3}{6\Omega}.$$

A. C. L. WILKINSON.

Flexion in Polar Coordinates.

(1) The conditions for *concavity* or *convexity* in polar coordinates as given in Text books are misleading. The statement 'that a curve is concave or convex according as the expression $\left(u+\frac{d^2u}{d\theta^2}\right)$ is *positive* or *negative*' requires modification when the sign of the radius vector is taken into account. Thus when the radius vector is *negative*, the condition for *concavity* or *convexity* is that $\left(u+\frac{d^2u}{d\theta^2}\right)$ should be *negative* or *positive*. The proof is quite obvious.

As an example, consider the hyperbola $u=1-2\cos\theta$. Here $u+\frac{d^2u}{d\theta^2}=1$, and is *always* positive. But, the curve is *concave* with respect

to the origin only so long as u is positive; that is, so long as θ *does not lie* between $-\frac{\pi}{3}$ and $+\frac{\pi}{3}$. Between these limits the curve is *convex*.

(2) Again, for *inflexion*, it is not enough that $\left(u+\dfrac{d^2u}{d\theta^2}\right)$ changes its sign. The condition that $\left(\dfrac{du}{d\theta}+\dfrac{d^3u}{d\theta^3}\right)$ is not *zero* must also be satisfied. This may be seen as follows :—

If $u=f(\theta)$ denotes a curve, the tangent at any point α is written as $u=f(\alpha)\cos(\theta-\alpha)+f'(\alpha)\sin(\theta-\alpha)$. So that, at a point of inflexion, $\{f(\alpha+\delta\alpha)-f(\alpha)\cos\delta\alpha-f'(\alpha)\sin\delta\alpha\}$ changes its sign Hence, the conditions are (i) $f(\alpha)+f''(\alpha)$ changes its sign; but (ii) $\{f'(\alpha)+f'''(\alpha)\}$ does not.

M. T. NARANIENGAR.

Orthopole.

Let L, M, N be the projections of the angular points of a triangle ABC on a chord PP′ of the circumcircle, and D, E, F and D′,E′,F′, the projections of P and P′ on the sides of ABC. Consider directly similar figures described on DD′, EE′, FF′. D, E, F, and D′, E′, F′ being collinear triads of corresponding points, the intersection of the lines DEF and D′E′F′ is the director point. PD, PE, PF and P′D′, P′E′, P′F′ being concurrent triads of corresponding lines, the invariable points are at infinity in directions at right angles to the sides of ABC. The circles through APEF and AP′E′F′ intersect in L which is therefore the centre of similitude of similar figures on EE′ and FF′. Similarly M, N are the remaining centres of similitude. The triangle of similitude is in perspective with the invariable triangle the centre of prespective being the director point. Hence the perpendiculars through L, M, N on the sides of ABC concur at the point of intersection of the pedal lines of P and P′.

M. BHIMASENA RAO.

Face of the Sky for February and March.

Mercury attains its greatest elongation (W) on February 2. It is in conjunction with Uranus on February 10, with the Moon on February 27 and March 31. It is in superior conjunction with the Sun on March 20 and so, until that date, is a morning star.

Venus is an evening star but too near the Sun. It is in conjunction with the moon on March 2 and with Saturn on March 29.

Mars is a morning star. It is in conjunction with the Moon on February 25 and March 26 and with Uranus on March 11.

Jupiter also is a morning star and better situated for observation than Mars. It is in conjunction with the Moon on February 19 and March 19. It is stationary on March 1 and retrogrades the rest of the month.

Saturn is in conjunction with the Moon on February 5 and on March 5 and is rapidly approching the Sun.

Uranus is in conjunction with the moon on February 25 and on March 25.

V. RAMESAM.

SOLUTIONS.

Question 154.

(K. J. SANJANA, M.A.,):—Let $S \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy$, and subscripts to S indicate equal subscripts in each coefficient. Show that the solution of the system of equations $S_1=p_1$, $S_2=p_2$, $S_3=p_3$ depends on the solution of a certain cubic; and hence or otherwise derive a method of solving these equations.

Solution by A. C. L. Wilkinson and the Proposer.

Choose multipliers λ_1, λ_2, λ_3 so that

$\lambda_1 p_1+\lambda_2 p_2+\lambda_3 p_3=o$, and the discriminant of

$\lambda_1 S_1+\lambda_2 S_2+\lambda_3 S_3=o$ also vanishes.

The discriminant when expanded can be written.

$$\lambda_1^3 \Delta_1 + \lambda_2^3\Delta_2 + \lambda_3^3\Delta_3+\lambda_1^2\lambda_2\Theta_{12}+\lambda_1\lambda_2^2\Theta_{21}+\lambda_1^2\lambda_3\Theta_{13}+\lambda_1\lambda_3^2\Theta_{31}$$
$$+\lambda_2\lambda_3^2\Theta_{32}+\lambda_3\lambda_2^2\Theta_{23}=o,$$

where Δ_1, Δ_2, Δ_3 have their usual meanings and $\Theta_{12}=\Sigma a_1\frac{\partial\Delta_2}{\partial a_2}$, &c.

The linear & cubic equations for λ_1 λ_2 λ_3 give three sets of ratios $\lambda_1 : \lambda_1 : \lambda_3$.

Choose any one of these sets and factorize $\lambda_1 S_1+\lambda_2 S_2+\lambda_3 S_3=o$. and the question then reduces to the solution of two equations, such as

$$S_1=p_1$$
$$S_2=p_2$$

and a linear equation $u_1 x+v_1 y+w_1 z=o$; and these reduce to two homogeneous equations in the ratios $x: y: z$, *viz.*,

$$u_1 x+v_1 y+w_1 z=o$$
$$S_1 p_2-S_2 p_1=o.$$

Question 158.

(*S. Narayanan, B.A., L.T.,*): *Solve the differential equation Completely* :—

$$\sin^2 2x\frac{d^3y}{dx^3}+6 \sin 2x \frac{d^2y}{dx^2}+ 4 (2-\cos 2x)\frac{dy}{dx}=O.$$

Solution (I) by A. C. L. Wilkinson, (II) by the Proposer.

I. Consider the equation.

$$\sin^2 2x\frac{d^2y}{dx^2}+6 \sin 2x \frac{dy}{dx}+4 (2-\cos 2x) y=O.$$

Reduce to normal form by the substitution.

$$y=b\ e^{-\frac{1}{2}\int\frac{6dx}{\sin 2x}}=v\ e^{-\frac{3}{2}\log\tan x}=v\ (\tan x)^{-\frac{3}{2}}$$

It becomes

$$\frac{d^2v}{dx^2}+\left\{\frac{4\ (2-\cos 2\ x)}{\sin^2 2x}+\frac{6\cos 2x}{\sin^2 2x}-\frac{9}{\sin^2 2x}\right\}v=0.$$ [Forsyth the Diff. Equ. art. 59]

or $$\frac{d^2v}{dx^2}+\frac{2\cos 2x-1}{\sin^2 2x}v=o$$

Now (Forsyth Diff. Equ. art. 68) the equation

$$\frac{dP_1}{dx}-P_1^2=\frac{2\cos 2x-1}{\sin 2x}$$ has the obvious solution

$$P_1=-\frac{1}{\sin 2x}.$$

Transform, therefore, by the substitution $v=ze^{-\int\frac{dx}{\sin 2x}}=z\tan^{-\frac{1}{2}}x$ and we get

$$\frac{dz}{dx}-\frac{2}{\sin 2x}z=a$$

$$\therefore\ \frac{d}{dx}(z\cot x)=a\cot x$$

and $z\cot x=a\log\sin x+b$

$\therefore\ y\tan x=a\log\sin x+b$

$\therefore\ y=\int Y\ dx=a\int\cot x\log\sin x dx+b\int\cot x\ dx+c$

$=a(\log\sin x)^2+b\log\sin x+c$

Knowing the complete primitive, many methods could now be adopted to solve the equation ; *e.g.*, the equation.

$$\tan^2 x\frac{d^2Y}{dx^2}+\frac{3\tan x}{\cos^2 x}\frac{dY}{dx}+Y\frac{1+2\sin^2 x}{\cos^4 x}=o$$ is exact, and integrates into

$$\tan^2 x\frac{dY}{dx}+Y\frac{\tan x}{\cos^2 x}=a,$$ and the solution is now easily completed. Or again, the substitution $\log\sin x=z$ must reduce the differential equation to $\frac{d^3y}{dz^3}=0$, but either of these methods implies a knowledge of the answer that only the proposer could posses.

II. We have

$$y_3\sin^2 2\ x+6\ y_2\sin 2\ x+4\ y_1\ (2-\cos 2\ x)=0$$

Dividing by $4\cos^4 x$, we get

$$y_3\tan^2 x+3\ y_2\tan x\sec^2 x$$
$$+y_1\ (\sec^2 x+3\tan^2 x\sec^2 x)=0$$

Hence a first integral is

$$y_2 \tan^2 x + y_1 \tan x \sec^2 x = 2a$$

or,

$$y_2 \tan x + y_1 \sec^2 x = 2\, a \cot x$$

which gives

$$y_1 \tan x = 2\, a \log \sin x + b$$

or,

$$y_1 = 2\, a \cot x \log \sin x + b \cot x$$

Hence the primitive is

$$y = a\, (\log \sin x)^2 + b \log \sin x + c$$

Question 201.

(M. BHIMASENA RAO) :—P is a common point of the director circles of the system of conics inscribed in a quadrilateral whose diagonal triangle is ABC. Show that the pedal circle of P with respect to the triangle ABC touches the pedal circles with respect to the same triangle of isogonal conjugate points on lines through P perpendicular to the sides of the quadrilateral. Apply the theorem when P is (1) the orthocentre (2) the circum-centre of the triangle ABC.

Solution by S. Purushottam.

Reciprocate with respect to P.

The 4 vertices reciprocate into the 3 ex-centres and the incentre of the △ into which the harmonic triangle reciprocates. The pedal circle of P inverts into the circumcircle of this triangle and the other circle into the auxiliary circle of a conic circumscribed to the △ and having its major axis passing through an excentre or the incentre.

Reciprocate and invert with respect to this ex-or incentre.

The conic reciprocates into an inconic whose major axis passes through the circumcentre.

The circumcircle of the △ inverts into the N.P.C of the new △ and the other circle into the auxiliary circle of the inconic. Hence, the 2nd circle has become the pedal of a point whose join with its isogonal conjugate passes through the circumcentre.

Hence, the proposition is true.

Question 204.

(M. T. NARAANIENGAR) :—In Question 69 shew that the inscribed conic has the following properties :—

(i) If X, Y, Z be the points of contact, AX, DY, DZ are $\perp^r$ to BC, CA, AB, respectively.

(ii) The foci of conic are the intersections of the circles DXYZA and BIC.

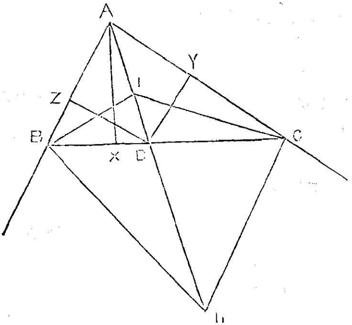
[**Question 69.**—(S. NARAYANAN, B.A., L.T.) :—If the bisector of the angle A of a triangle meets the base in D, shew that the lines joining B and C to the symmedian points of the triangles ABD, ACD respectively meet on AD.

Hence shew that this point is the centre of the conic which is inscribed in the triangle ABC and which has double contact along AD with the conic which passes through the points of contact with the side of the inscribed circle and the escribed circle corresponding so A.]

Solution by S. Purushottam.

(1) Let PQR,P′Q′R′ be inscribed and escribed circles. It is readily seen that if two conics have double contact, the polar of any point on the chord of contact is the same for the two conics.

Now, the projections of the harmonic range AI DI_1, on the sides are also harmonic ranges. Therefore, the polar of A with respect to the conic PQR P′Q′R′ passes through Y and Z. That is YZ is the polar of A with respect to the conic PQR P′Q′R′ ; and since A is a point on the chord of contact of this conic with the proposed in conic YZ must also be the polar of A with respect to this in conic. Hence, the inconic touches AB, AC at Z and Y. It touches BC at X, since $AZ . BX . CY = AY . AX \cot B . DY \cot C = AY . AX \cot C . DZ \cot B = AY . CX . BZ$.



(2) Again, AD is a principal axis of the inconic, because it is perpendicular to its conjugate line YZ ; and it cannot be the major axis, since, in that case, the foci must be the double points of the overlapping involution (AD, II_1,). Hence, AD is the minor axis of the inconic and its foci lie on the circle AXYZ by a well known theorem. Further, as the confocal through B has BI, BI_1, for tangent and normal and II_1 being its minor axis ; therefore the foci must also lie on the circle BII_1 , that is, on the circle BIC.

The first part suggests a geometrical proof of Q. 69. For XZ and AD are antiparallel, with respect to BA and BC. Hence, the line joining B to the middle point of ZX is the symmedian through B of the △ ABD. Hence, this line passes through the centre of the conic. Similarly, the symmedian through C of the △ ACD passes through the centre. In other words the centre is the isogonal conjugate of the mid-point of AD, which is obviously a point in AD itself.

Additional solution by M. Bhimasena Rao.

Question 216.

(K. J. SANJANA, M.A.) :—A conic touches the sides of a triangle and passes through two isogonally conjugate points; prove that the pole of the line joining the points is one of four fixed points.

Solution by S. Purushottam.

By Q. 39 the co-ordinates of any point are the geometric means of the co-ordinates of the points of contact of tangents from it to the in-conic. Hence, the pole of the join of the isogonal points is the in centre or an excentre of the triangle.

Additional Solutions by J. C. Swaminarayan, M.A., and S. Narayan B.A., L.T.

Question 218.

(J. C. SWAMINARAYANAN, M.A.) :—Shew that the area of the loop of the curve, $3y^2(x\sqrt{2}+a)=x^2(3a-x\sqrt{2})$ is equal to the area between the same curve and its asymptote.

Solution by A. V. K. Menon and the proposer

Area of the loop $=2\int_0^{\frac{3a}{\sqrt{2}}} y\,dx$, evidently and area of the portion between the curve and the asymptote

$$=2\int_0^{\frac{-a}{\sqrt{2}}} y\,dx.$$

Now, $\int y\,dx=\frac{1}{\sqrt{3}}\int x\sqrt{\frac{3a\sqrt{x}\sqrt{2}}{u\sqrt{2}+a}}\,dx.$

Put $x\sqrt{2}=a(3-4\sin^2\phi)$.

$$\therefore \int y\,dx = \frac{4a^2}{\sqrt{3}} \int \sin^2\phi\,(4\sin^2\phi - 3)\,d\phi.$$

$$= \frac{2a^2}{\sqrt{3}}[\tfrac{1}{4}\sin 4\phi - \tfrac{1}{2}\sin 2\phi].$$

$$= \frac{a^2}{2\sqrt{3}}(\sin 4\phi - 2\sin 2\phi).$$

When $x = \frac{3a}{\sqrt{2}}$, $\phi = o$; when $x = o$, $\phi = \frac{\pi}{3}$ and when $x = -\frac{a}{\sqrt{2}}$, $\phi = \frac{\pi}{2}$

$$\therefore \int_o^{\frac{3a}{\sqrt{2}}} y\,dx = a^2 \,[\sin 4\phi - 2\sin 2\phi]_{\frac{o}{4}} = \frac{3a^2}{4} \text{ and } \int_o^{\frac{-a}{\sqrt{2}}} y\,dx = \frac{a^2}{2\sqrt{3}}$$

$$[\sin 4\phi - 2\sin 2\phi]_{\frac{4}{3}}^{\frac{4}{2}} = \frac{3a^2}{4}.$$ Hence the result follows.

Question 221.

(S. Narayan, B.A., L.T.) :—ABC is a triangle whose medial triangle is DEF. The join of the incentres of ABC, DEF cuts DF, DE in K & L. If BK, CL meet AC, AB respectively in P&Q shew that PQ touches the incircle and the nine-point circle of ABC.

(*I*) *Solution by R. Ramachandra Rao and* (*II*) *by M. Bhimasena Rao.*

I. The co-ordinates of I′ the centre of DEF are $a\,(b-c)$, &c. and of I are 1, 1, 1.

Hence the equation of II′ is $a\,\alpha\,(b-c) +$ &c. $= 0$.

Hence CQ is $a\,\alpha\,(c-a) = b\,\beta\,(c-b)$ and BQ is $a\,\alpha\,(b-a) = c\gamma\,(b-c)$

$\therefore$ the equation of PQ is $\frac{a\,\alpha}{b-c} + \frac{b\,\beta}{c-a} + \frac{c\,\gamma}{a-b} = 0$ which is known to be the tangent at the common point of the in and nine point circles

II. It is evident that KL is the diagonal line of the quadrilateral PCBQ. ABC and DEF being homothetic, KL passes through the homothetic centre which is the median point G. Therefore we have to show that the common tangent of the in and the nine point circles touches the in-conic of ABC whose centre is G *i.e.*, the Steiner ellipse of DEF.

The common tangent is the line $\frac{a\,\alpha}{b-c} + \frac{b\,\beta}{c-a} + \frac{c\,\gamma}{a-b} = 0.$

This touches the Steiner ellipse $(a\alpha)^{\frac{1}{2}}+(b\beta)^{\frac{1}{2}}+(c\gamma)^{\frac{1}{2}}=0$ at the point $\frac{(b-c)^2}{a}, \frac{(c-a)^2}{b}, \frac{(a-b)^2}{c}$.

The common tangents at the remaining Feuerbach points are $\frac{a\,\alpha}{b-c}+\frac{b\,\beta}{c+a}-\frac{c\,\gamma}{a+b}=0$ etc. which shows that the theorem is true for the excentres also.

Question 227.

(R. N. Apte M.A., F.R.A.S) :—Find the eliminant of $\frac{m_1}{x-a_1}+\frac{m_2}{x-a_2}+\frac{m_3}{x-a_3}=0, \frac{n_1}{x-b_1}+\frac{n_2}{x-b_2}+\frac{n_3}{x-b_3}=0$ and deduce that of

$$\frac{m_1}{x-a_1}+\frac{m_2}{x-a_2}+\frac{m_3}{x-a_3}+\frac{m_4}{x-a_4}=0,$$ and a similar equation with n, b. written for m, a. Generalise the result.

(Dixon's Determinants).

Solution by 'Unity'

In the case of three fractions we have, after reduction

$$x^2\Sigma m_1-x\Sigma m_1\overline{a_2+a_3}+\Sigma m_1a_2a_3=0$$

$$x^2\Sigma n_1-x\Sigma n_1\overline{b_2+b_3}+\Sigma n_1b_2b_3=0$$

Eliminating x by Sylvester's method we have.

$$0=\begin{vmatrix} \Sigma m_1 & \Sigma m_1\overline{a_2+a_3} & \Sigma m_1a_2a_3 & 0 \\ 0 & \Sigma m_1 & \Sigma m_1\overline{a_2+a} & \Sigma m_1a_2a_3 \\ 0 & \Sigma n_1 & \Sigma n_1\overline{b_2+b_3} & \Sigma n_1b_2b_3 \\ \Sigma n_1 & \Sigma n_1\overline{b_2+b_3} & \Sigma n_1b_2b_3 & 0 \end{vmatrix}$$

Similarly, for the other cases.

Question 228.

(N. Durai Rajan) :—Given the centroid and the Fermat point of a triangle, find the locus of the angular points. [The Fermat point is that at which the sides subtend equal angles].

Remarks by M. Bhimasena Rao and S. Purushottam.

Through the Fermat point F draw three lines FA, FB, FC making equal angles with one another. Take any point A′ on FA, join A′ to the centroid G and produce A′G to D such that GD $=\frac{1}{2}$ A′G. Through D we can draw a line B′C′ meeting FB, FC in B′, C′ such that D is the mid point of B′C′. Then the triangle A′B′C′ has G for its centroid and F for its Fermat point. The angular points do not move on any particular loci.

Question 230.

(P. V. Seshu Aiyar, B.A. L.T.) :—

If $\frac{a}{b} \equiv q_1 + \frac{1}{q_2+} \frac{1}{q_3+} \cdots\cdots + \frac{1}{q_{2n-1}} + \frac{1}{q_{2r}}$

Show that the denominator of the fraction

$$q_1 + \frac{1}{q_2+} \cdots\cdots + \frac{1}{q_{2n-1}} + \frac{t}{q_{2n}t \pm c}$$

is $bt \pm cq$, where q is the denominator of the penultimate convergent.

Solution by S. Narayanan and R. Kausikar.

The last element in the given fraction may be written...$+\frac{1}{q_{2n}}+\frac{\pm c}{t}$

Here according to the usual law,

$$\begin{aligned} Q_{2n+1} &= q_{2n+1} \cdot Q_{2n} + r_{2n+1}\, Q_{2n-1} \\ &= t.b \pm cq \\ &= bt \pm cq. \end{aligned}$$

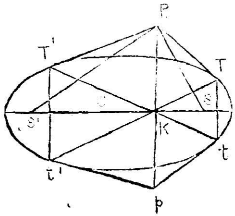
[where Q_{2n-2}, Q_{2n}, Q_{2n+1} denote the denominators of successive convergents and p_{2n+1}, q_{2n+1} are the members of the $2\,n+1^{th}$ element]

Question 233.

(S. Narayanan, B.A., L.T.) :—S and S′ are the foci of an ellipse whose eccentricity is greater than $\frac{1}{\sqrt{2}}$. P is a variable point on the circle on SS′ as diameter and PT, PT′ are tangents to the ellipse. Shew that the circumcentre of the triangle PTT′ lies on the major axis of the ellipse.

Solution (I) by S. Purushottam, (II) by R. Ramachandra Rao, and R. Srinivasan, M.A.

(I) Let p,t,t' be the reflexions of P,T,T′ in SS′ and let Tt′ T′t cut in K, which evidently lies on SS and Pp.



Now, SP,S′P which bisect the angles TST′ and TS′T′ are, by hypothesis, at right angles. Therefore, the angles STS′ and ST′S′ are supplementary. Thus the points STS′t′ are concyclic and TK . Kt′ = SK.. KS′ = PK . Kp. Hence, also, Pp t′T is a circle which, by symmetry passes through T′ and t., In other words, the circle PTT′ passes through p′ t t′ and has obviously its centre on the major axis

(II.) If the eccentric angles of T,T′ be ϕ, ϕ' the point P is easily found to be $x=a\cos\frac{1}{2}(\phi+\phi')\sec\frac{1}{2}(\phi-\phi')$, $y=b\sin\frac{1}{2}(\phi+\phi')$. $\sec\frac{1}{2}(\phi-\phi')$. The centre of the circle PTT′ is (h, k) such that $2k-b\sin\frac{1}{2}(\phi+\phi')\sec\frac{1}{2}(\phi-\phi')=\frac{a^2-b^2}{b}\frac{\cos\phi'-\cos\phi}{\cot\phi-\cot\phi'}$, so that the centre lies on the major axis, if $k=0$. That is, if $(a^2-b^2)\sin\phi\,.\sin\phi'=b^2$. (i)

But P lies on the circle $x^2+y^2=(a^2-b^2)$. Substituting the co-ordinates of P and simplifying, this reduces to $a^2\cos^2\frac{1}{2}(\phi+\phi')+b^2\sin^2\frac{1}{2}(\phi+\phi')=(a^2-b^2)\cos^2\frac{1}{2}(\phi-\phi')$

(i.e.) $a^2\{\cos^2\frac{1}{2}(\phi-\phi')-\cos^2\frac{1}{2}(\phi+\phi')\}=b^2\{\sin^2\frac{1}{2}(\phi+\phi')+\cos^2\frac{1}{2}(\phi-\phi')\}$

(i.e.) $a^2\sin\phi\,.\sin\phi'=b^2(1-\sin\phi\sin\phi')$, which is the same as (i)

Question 234.

(UNITY) :—Give an analytical proof for the following :

Three tangents to a parabola at the points A B C form an equilateral triangle. Prove that the ellipse which touches the triangle A,B,C at the middle points of the sides passes through the focus of the parabola.

Solutions (I) by M. Bhimasena Rao, and (II) by R. Ramachendra Rao.

Let l, m, n denote the lengths of the sides of the triangle L M N formed by the tangents at A, B, C.

Referred to the triangle LMN, let $(p\alpha)^{\frac{1}{2}}+(q\beta)^{\frac{1}{2}}+(r\gamma)^{\frac{1}{2}}=0$ be the equation of the parabola. AL, BM, CN intersect at $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$. Its centre which is the point of contact of the curve with $l\alpha+m\beta+n\gamma=0$, is $\frac{p}{l^2}, \frac{q}{m^2}, \frac{r}{n^2}$. Its focus, the isogonal oonjugate of the centre is $\frac{l^2}{p}, \frac{m^2}{q}, \frac{n^2}{r}$. When LMN is equilateral $l=m=n$ and the focus is $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$ showing that that AL, BM, CN will then intersect at the focus ... ... (1)

Referred to the triangle ABC let $\frac{f}{\alpha}+\frac{g}{\beta}+\frac{h}{\gamma}=0$ be the equation of the parabola. Since the line at infinity touches it $(af)^{\frac{1}{2}}+(bg)^{\frac{1}{2}}+(ch)^{\frac{1}{2}}=0$ ... (2)

Now tangent at B is $f\gamma+h\alpha=0$
Tangent at C is $f\beta+g\alpha=0$
∴ L is the point $-f, g, h$.

Hence AL, BM, CN intersect at f, g, h. Therefore by (2)

AL, BM, CN intersect on the ellipse $(a\alpha)^{\frac{1}{2}}+(b\beta)^{\frac{1}{2}}+(c\gamma)^{\frac{1}{2}} = o$ (3) which tonches the sides of ABC at the middle points

The reqnired result follows from (1) and (3).

(1I). Let ABC be the pts. l, m, n

$$\text{Then } BC^2=\left(\frac{p}{l^2}-\frac{p}{m^2}\right)^2+4\left(\frac{p}{l}-\frac{p}{m}\right)^2$$

$$=\frac{p^2}{l^4m^4}(l-m)^2\{(l+m)^2+4l^2m^2\}$$

$$\text{The } \perp^r \text{ from } (x,y) \text{ on BC is } \frac{2xlm-y(l+m)+2p}{\sqrt{\{l+m)^2+4l^2m^2\}}}$$

$$\therefore BC\times\perp \text{ on BC is } [2xlm-y(l+m)+2p]\frac{p}{l^2m^2}(l-m)$$

The equation of the midellepse is $\sqrt{BC\times\perp \text{ on } BC}+\&=0$

This passes through the focus ($x=p$, $y=0$), if

$$\sqrt{(2plm+2p)\frac{p}{l^2m^2}(l-m)}+c=0$$

(*i.e.*) if $n\sqrt{(l-m)(1+lm)}+\&c=0$

If the $\triangle$ of tangents be equilateral

$$l-m=\tan 60^\circ(1+lm),\ \&c,$$

Therefore the condition reduces to n $(l-m)+\&c=0$, which is an identity,

Additional solutions by S. Narayanan B.A., L.T., and S. Purushottam.

Question 235.

(UNITY) :—Show that through any point on an ellipse six osculating circles of an ellipse pass and that their centres are on another conic. What does this become when the point is on the ellipse?

This is completely solved in Casey's Analytical conics page 316.

Question 236.

(J. C. SWAMINARAYANAN, M. A) :—Find the envelope of the straight line.

$$x\sqrt{c^2-\lambda^2}-\lambda y+c^2$$

$$-\frac{c}{2}\sqrt{c^2-\lambda^2}\log\frac{c+\sqrt{c^2-\lambda^2}}{c-\sqrt{c^2-\lambda^2}}=0$$

where λ is any arbitrary parameter

Additional solution by M. Bhimasena Rao.

It is easily seen that $\frac{1}{2}\log\frac{c+\sqrt{c^2-\lambda^2}}{c-\sqrt{c^2-\lambda^2}}=\sinh^{-1}\frac{\sqrt{c^2-\lambda^2}}{\lambda}=\theta$, say.

Then $\sinh\theta=\frac{\sqrt{c^2-\lambda^2}}{\lambda}$ and $\cosh\theta=\frac{c}{\lambda}$

The given equation may now be written as

$$x\sinh\theta-c\,\theta\sinh\theta=y-c\cosh\theta \text{——}(1)$$

Differentiating with respect to θ we have

$$x\cosh\theta-c\,\theta\,.\cosh\theta-c\sinh\theta=-c\sinh\theta$$

$$\therefore\ x-c\,\theta=0 \text{ i.e. } \theta=\frac{x}{c}.$$

Substituting in (1) we have $y=c\cosh\frac{x}{c}$ which is the required envelope. (The chainette).

Question 237.

(M. Bhimasena Rao): Show that the circum and the nine points circles of a triangle intersect in isotomic conjugate points with respect to the medial triangle of the given triangle and generalise by projection.

Solution by S. Purushottam and others

Take the medial triangle as the triangle of reference. The equation in areal co-ordinates of any circle is

$\Sigma a^2\beta\gamma+(\alpha+\beta+\gamma)(l\alpha+m\beta+n\gamma)=0$

The circle

$\Sigma a^2\beta\gamma+(\alpha+\beta+\gamma)(a^2\alpha+b^2\beta+c^2\gamma)=0$

passes through the points $(-1, 1, 1)$, $(1, -1, 1)$and $(1, 1, -1)$ *i.e.* is the circumcircle of the original triangle.

The isotomic transformation of the straight line whose equation is

$\Sigma a^2\alpha=0$

is $\Sigma a^2\beta\gamma=0$, which is the N.P. circle,

because the points (α,β,γ) and $\left(\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}\right)$ are isotomic conjugates.

Hence the N.P.C. and the circumcircle intersect in isotomic conjugates with respect to the medial triangle.

Question 239.

(A. V. K. MENON, M.A.) :—Prove that when $|k|<1$

$$\int_0^{\frac{\pi}{2}} \frac{\cos^m\theta \sin^n\theta \, d\theta}{(1-k\sin^2\theta)^{\frac{1}{2}}} = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{m+n+1}{2}\right)} \int_0^{\frac{\pi}{2}} \frac{\cos^{m+n}\theta \, d\theta.}{(1-k\sin^2\theta)^{\frac{n+1}{2}}}$$

Solution by R. Srinivasan and S. Purushottam.

Whatever be the value of n, we have $\Gamma(n).n(n+1)(n+2)\ldots\ldots(n+m)=\Gamma(n+m)$. Now, the coefficient of k^r in the expansion of $(1-k\sin^2\theta)^{-\frac{n+1}{2}}$ is $\dfrac{\Gamma\left(\frac{n+qr+1}{2}\right)\sin^{2r}\theta}{\Gamma\left(\frac{n+1}{2}\right)}$ and the coefficient of k^r in the expansion of $(1-k\sin^2\theta)^{-\frac{1}{2}}$ is $\dfrac{\Gamma\left(\frac{2r+1}{2}\right)\sin^{2r}\theta}{\Gamma(\frac{1}{2})}$

Hence the left hand side expression

$$=\int_0^{\frac{\pi}{2}} \sum_{r=0}^{r=\infty} k^r\cos^m\theta \sin^{n+2r}\theta \, d\theta \times \frac{\Gamma\left(\frac{2r+1}{2}\right)}{\Gamma(\frac{1}{1})}$$

$$=\sum k^r \frac{\Gamma\left(\frac{2r+1}{2}\right)}{\sqrt{\pi}} \int_0^{\frac{\pi}{2}} \mathrm{Cos}^m\theta \sin^{n+2r}\theta \, d\theta$$

$$=\sum k^r \frac{\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{m+1}{1}\right)\Gamma\left(\frac{n+2r+1}{2}\right)}{2\sqrt{\pi}\,\Gamma\left(\frac{m+n+2r+2}{2}\right)}$$

The right hand side expression equals

$$\int_0^{\frac{\pi}{2}} \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\Pi}\,\Gamma\left(\frac{m+n+1}{2}\right)} \sum_{r=0}^{r=\infty} k^r \frac{\Gamma\left(\frac{n+2r+1}{-}\right)}{\Gamma\left(\frac{n+1}{2}\right)} \cos^{m+n}\theta \sin^{2r}\theta \, d\theta$$

$$=\sum k^r \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+2r+1}{2}\right)}{\sqrt{\Pi}\,\Gamma\left(\frac{m+n+1}{2}\right)} \int_0^{\frac{\pi}{2}} \cos^{m+n}\theta \sin^{2r}\theta \, d\theta$$

$$=\sum k^r \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+2r+1}{2}\right)}{\sqrt{\Pi}\,\Gamma\left(\frac{m+n+1}{2}\right)} \frac{\Gamma\left(\frac{m+n+1}{2}\right)\Gamma\left(\frac{2r+1}{2}\right)}{2\Gamma\left(\frac{m+n+2r+2}{2}\right)}$$

$$=\sum k^r \frac{\Gamma\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+2r+1}{2}\right)}{2\sqrt{\Pi}\,\Gamma\left(\frac{m+n+2r+2}{2}\right)}$$

Hence the result.

Question 241.

(R. SRINIVASAN, M.A.) :—Show that

$$\sum_{n=o}^{n=a} \frac{1-n(n-1)x^2}{x^2+\{1+n(n-1)x^2\}^2}=\frac{a}{1+a^2x^2}$$

Solution by R. Kausikar.

$\tan^{-1} nx-\tan^{-1}(n-1)x=\tan^{-1}\frac{x}{1+n(n-1)x^2}$; give n successive values from $n=a$ to $n=o$ and add ; $\tan^{-1} ax=\sum_{n=c}^{n=o}\tan^{-1}\frac{x}{1+n(n-1)x^2}$:

differentiating with respect to x.

we get $\sum_{n=o}^{n=a} \frac{1-n(n-1)x^2}{x^2+\{1+n(n-1)x^2\}^2}=\frac{a}{1+a^2x^2}$

Additional solution by Unity

Question 242.

(N. DURAI RAJAN) :—Shew that at an ordinary point P on a curve of the nth degree, the following relation among the radii of curvature of the curve and the polar curves of P, holds :—

$(n-1)\rho=(n-2)\rho_1=(n-3)\rho_2=\ldots\ldots=1.\ \rho_{n-2}$, where ρ_r is the radius of curvature of the rth polar at P.

Solution by R. Srinivasan, M.A. and several others.

If the point P and the tangent at it be taken as the origin and the axis of y, the equation to the curve can be put in the form,

$$u_n+u_{n-1}+u_{n-2}+\ldots\ldots+u_2+x=0 \qquad \text{(A)}$$

where u_r represents a homogeneous expression of ths r^{th} degree in x and y.

Its polar curve with regard to the origin is

$$u_{n-1}+2\ n_{n-2}+3\ u_{n-3}+\ldots\ldots+(n-2)\ u_2+(n-1)\ x=0 \qquad \text{(B)}$$

Let u_2 be of the form $ax^2+2hxy+by^2$.

Then the radius of curvature at P for (A)

is $\underset{x=y=o}{\text{Lt}} \dfrac{y^2}{2x}=-\dfrac{1}{2b}$

and that for (β) is $\dfrac{n-1}{n-2}\cdot\left(-\dfrac{1}{2b}\right)$

$$\therefore \rho_1=\frac{n-1}{n-2}\rho$$

i.e. $(n-1)\ \rho=(n-2)\ \rho_1$

Hence, extending this result, we get

$(n-1)\ \rho=(n-2)\rho_1=(n-3)\rho_2=\ldots$

Question 243.

(O. N.) :—Show that 37 is the only prime number less than 1200 which is such that when multiplied by 345, the product has 7 in the hundred's place and 6 in the ten's place.

Solution by R. Kausikar and others.

				3	4	5
			d	*c*	*b*	*a*
			.	*f*	*e*	5
		.	.	*h*	*g*	
	.	.	.	*i*		
.	.	.	.			
				7	6	5

As the number is prime a is odd but since g is o or 5, e is 6 or 1 ; of 1, 3, 5, 7, 9, (the possible valves of a) we find that only 7 satisfies this condition. W ehave now $e=1$; $g=5$; $f=4$. Since $g=5$, b is odd and $h+i=3$ or 13 $\therefore$ $h=8$, $i=5$, or $h=3$, $i=0$; but no odd value of b gives 8 for h $\therefore$ $h=3$ and $i=0$: $\therefore$ c and $\therefore d$ c is even and less than 12 since the number is less than 1200 $h=3$ only if $b=3$ $\therefore$ the possible numbers are 237 $(=3\times79)$ 437 $(=19\times23)$. 637 $(=7\times91)$, 837 $(=3\times279)$; 1037 $(=17\times61)$ none of which are prime. $\therefore$ $dc=o$ and the required No. is 37.

QUESTION FOR SOLUTION.

259. (T. RAJARAM RAO, B.A., B.L.) :—AB is a finite straight line and P is a fixed point on AB produced. O is another fixed point on the line which bisects AB at right angles. An ellipse is described passing through A and B with O as one of its foci. If PQ is the tangent to the ellipse, find the locus of Q. What difference will it make in case the curve is a hyperbola?

260. (S. RAMANUJAM *communicated by* MR. P. V. SESHU AIYAR) :— Show without using calculus, that

$$\frac{3}{2}\log 2 = 1 + \frac{2}{4^3-4} + \frac{2}{8^3-8} + \frac{2}{12^3-12} + \quad \ldots \quad \ldots \textit{ ad inf.}$$

261. (S. RAMANUJAM, through MR. P. V. SESHU AIYAR):—Show that

$$(a) \quad \left(1+\frac{1}{1^3}\right)\left(1+\frac{1}{2^3}\right)\left(1+\frac{1}{3^3}\right) \quad \ldots \quad \ldots \textit{ad inf} = \frac{1}{\pi}\cosh\frac{\pi\sqrt{3}}{2}$$

$$(b) \quad \left(1-\frac{1}{2^3}\right)\left(1-\frac{1}{3^3}\right)\left(1-\frac{1}{4^3}\right) \quad \ldots \quad \ldots \quad = \frac{1}{3\pi}\cosh\frac{\pi\sqrt{3}}{2}$$

and prove from first principles that $(b) = \frac{1}{3}(a)$.

262. (M. BHIMASENA RAO.) :—DEF is a pedal triangle of a triangle ABC. If AD, BE, CF be concurrent, show that the circle DEF intersects the ninepoint circle of ABC on a tangent to the maximum inscribed ellipse of ABC. Deduce immediately the truth of question 248, (1) by Mr. S. Narayanan, B.A., L.T.

263. (M. BHIMASENA RAO) .—A variable triangle ABC is inscribed in a conic X and circumscribed about another conic Y. If the isogonal conjugate of X with respect to ABC in any one of its positions touches Y, the isogonal conjugate of X with respect to ABC in any other position also touches Y.

264. (SELECTED) :—Plant 21 trees in a park so that they should form 12 straight rows with 5 trees in every row.

265. (ZERO) :—The corners of a regular polygon of $2n$ sides are joined to the centre O, and any transversal cuts the joining lines in A, B, C, ...Prove that $\Sigma(OA^2) = n^2 . OM^2$, where M is the *mean centre* of A, B, C,......

266. (N. DURAI RAJAN) :—Describe a rectangular hyperbola so as to touch 4 concentric circles.

267. (N. DURAI RAJAN) :—A straight line cuts a plane cubic curve in 3 pts. P, Q, R. and the mean centre of P, Q, R is on the cubic. Find the envelope of the line PQR.

268. (R. Srinivasan M.A.,) :—Show that the set of equations.

$$\frac{x^4}{a^{\frac{5}{4}}b^{\frac{3}{4}}}+\frac{y^4}{a^{\frac{3}{4}}b^{\frac{5}{4}}}=mz^2;$$

$$\frac{x^{\frac{3}{2}}}{(az)^{\frac{1}{4}}}=x+y=\frac{y^{\frac{3}{2}}}{(bz)^{\frac{1}{4}}}$$

is possible for real values of x, y, z only if m is greater than 512.

269. (N. Sankara Aiyar B.A.,) :—If A, B, C, D be the feet of four normals drawn from a point M to an ellipse whose centre is O, the perpendiculars from the circumcentres of the triangles formed by any three of the points A, B, C, D upon the common chord of the ellipse and the osculating circle at the fourth point bisects the line OM.

270. (S. Narayanan B.A., L.T.,) :—Shew that the locus of the orthopole of a straight line passing through the incentre of a triangle ABC is a conic inscribed in the excentral triangle of the medial triangle of ABC having its centre at the circumcentre of that triangle.

271. (M. T. Naraniengar M A.,) :—AB, A′B′ are two chords of a circle with centre O. A′B, AB′ intersect in S and the diameter bisecting AB meets the perpendicular to A′B′ through S in U. Shew that U is the circumcentre of ASB. Also, if V be the inverse of U with respect to the circle O, prove that the triangles AVB, A′OB′ are similar.

272. (M. T. Naraniengar M.A.) :—If S, H be the foci of an inconic of a triangle ABC, shew that the circumcentres of ASB, AHB are inverse points with respect to the circle ABC.

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